

Fall Quarter 2011 UCSB Physics 225A Homework 2

General hint

A useful trick to simplify neutrino oscillation problems is to make use of unitarity, for example:

$$U_{ei}U_{ej}^* + U_{\mu i}U_{\mu j}^* + U_{\tau i}U_{\tau j}^* = \delta_{ij}$$

Then for $i \neq j$ multiply by $U_{ei}^*U_{ej}$ to get

$$|U_{ei}U_{ej}^*|^2 + U_{ei}^*U_{ej}(U_{\mu i}U_{\mu j}^* + U_{\tau i}U_{\tau j}^*) = 0$$

$$U_{ei}^*U_{ej}(U_{\mu i}U_{\mu j}^* + U_{\tau i}U_{\tau j}^*) = -|U_{ei}U_{ej}^*|^2$$

Another useful trick is

$$\sum_i U_{\alpha i}^* U_{\beta i} = \delta_{\alpha\beta}$$

Take the square of this:

$$\sum_{ij} U_{\alpha i}^* U_{\beta i} U_{\alpha j} U_{\beta j}^* = \delta_{\alpha\beta}$$

$$\sum_i U_{\alpha i}^* U_{\beta i} U_{\alpha i} U_{\beta i}^* + 2 \sum_{i < j} U_{\alpha i}^* U_{\beta i} U_{\alpha j} U_{\beta j}^* = \delta_{\alpha\beta}$$

Rearranging:

$$\sum_i U_{\alpha i}^* U_{\beta i} U_{\alpha i} U_{\beta i}^* = \delta_{\alpha\beta} - 2 \sum_{i < j} U_{\alpha i}^* U_{\beta i} U_{\alpha j} U_{\beta j}^*$$

You can do similar things interchanging the roles of rows and columns.

• Problem 1

(a) Consider a pion of energy E_π in the LAB frame. Find the LAB energy of the neutrino (E_ν) in the decay $\pi^+ \rightarrow \mu^+ \nu_\mu$ as a function of the LAB angle θ that the emitted neutrino makes with the original flight direction of the π^+ .

(b) Plot E_ν for E_π between 2 and 20 GeV in the case $\theta = 0$ and $\theta = 15$ mrad.

You should find that in the case $\theta = 15$ mrad the range of E_ν is much reduced. The next-generation neutrino experiments, T2K in Japan and NuA in the US, will place their detector off-axis by a small amount. They will use this trick to obtain a beam with a narrow spread of E_ν . This has several advantages, *e.g.*, the energy and baseline can be tuned to be at a maximum of the oscillations (now that the values of Δm^2 are well known).

• **Problem 2**

As an exercise in the usage of natural units, show that the (dimensionless) quantity $\Delta = \Delta m^2 L / (4E)$ that appears in the theory of neutrino oscillations is $\Delta = 1.27 \Delta m^2 (\text{eV}^2) L (\text{Km}) / E (\text{GeV})$.

• **Problem 3**

Reactors are a source of $\bar{\nu}_e$. The next generation of neutrino reactor experiments will attempt to measure the parameter $\sin^2 2\theta_{13}$ of the neutrino mixing matrix. They will measure the probability of $\bar{\nu}_e$ disappearance a distance L from the reactor. Show that this probability is given by

$$\mathcal{P} = \sin^2 2\theta_{13} \sin^2 \Delta_{31} + \cos^4 \theta_{13} \sin^2 2\theta_{12} \sin^2 \Delta_{21}$$

where the θ_{ij} 's are the angles of the neutrino mixing matrix and $\Delta_{ij} = \frac{1}{4}(m_i^2 - m_j^2)L/E$. You can make the approximation that $\Delta_{31} = \Delta_{32}$. Next, verify that for the typical $E = 3 \text{ MeV}$ and $L = 1.5 \text{ Km}$, Δ_{31} is close to maximal (90°) and Δ_{21} is quite small. Take $|\Delta m_{21}^2| = |m_2^2 - m_1^2| = 8.0 \times 10^{-5} \text{ eV}^2$ and $|\Delta m_{31}^2| = |m_3^2 - m_1^2| = 2.4 \times 10^{-3} \text{ eV}^2$.

These new experiments, Daya Bay in China and Double Chooz in France, propose to reach sensitivities for $\sin^2 2\theta_{13}$ of order 0.01 and 0.03 respectively. This means that they will be looking for disappearance probabilities at the percent level or below. In order to achieve this goal, the flux of $\bar{\nu}_e$ from the reactor needs to be known extremely well. Both experiments plan to actually deploy two or more detectors. One (or more) detectors will be placed near the reactor, to measure the flux near $L = 0$, and other detectors will be placed far from the reactor ($L = 1 - 2 \text{ Km}$) to measure the flux near the oscillation maximum.

• **Problem 4**

(a) Show that the effective Majorana neutrino mass for neutrinoless double beta decay, $m_{\beta\beta} \equiv |\sum m_i U_{ei}^2|$ depends on the Majorana phases ϕ_1 and ϕ_2 given in equation (40) of hep-ph/0506165, as well as the CP violating phase δ .

(b) We know from oscillation experiments that $|\Delta m_{21}^2| \sim 8.0 \times 10^{-5} \text{ eV}^2$, $|\Delta m_{31}^2| \sim |\Delta m_{32}^2| \sim 2.4 \times 10^{-3} \text{ eV}^2$, $\theta_{32} \sim 45^\circ$, $\theta_{12} \sim 34^\circ$, and $\theta_{13} < 10^\circ$.

We also know from tritium β decay experiments that neutrino masses are lighter than about 2 eV. However, we know nothing about ϕ_1 , ϕ_2 , and δ .

Use these information to show that in the case of the normal mass hierarchy the unknown parameters of the neutrino sector could conspire to make $m_{\beta\beta}$ vanishingly small. (You do not have to give an absolute proof, unless you really want to. A handwaving argument is enough). On the other hand, show that in the case of the inverted mass hierarchy

$$m_{\beta\beta} \sim m_0 \sqrt{1 - \sin^2 2\theta_{12} \sin^2 \frac{\Delta\phi}{2}} \geq m_0 \cos 2\theta_{12} \geq \sqrt{|\Delta m_{31}^2|} \cos 2\theta_{12}$$

where $m_0 = \frac{1}{2}(m_1 + m_2)$ and $\Delta\phi = \phi_1 - \phi_2$.

Note: Here I made some approximation justified by the actual values of the neutrino parameters that have been measured, i.e., I have taken $\cos \theta_{13} \sim 1$ and I have neglected $\frac{1}{2}|m_2 - m_1|$ compared to m_0 in the case of the inverted mass hierarchy. (Convince yourself that this is a reasonable thing to do!).

This inequality is important because it provides a theoretical “floor” to $m_{\beta\beta}$ and therefore to the rate for $0\nu\beta\beta$ decay, since this rate is proportional to $m_{\beta\beta}^2$. If we were able to set an upper limit on the $0\nu\beta\beta$ decay rate below this theoretical floor, we would be able to conclude that neutrinos are **not** Majorana particles. Unfortunately the argument holds only for the inverted mass hierarchy.

• Problem 5

The amplitude for $\nu_\alpha \rightarrow \nu_\beta$ in the general case of N neutrino generations was given in class and it is also given in equation (10) of hep-ph/0506165.

(a) Show that the probability for $\nu_\alpha \rightarrow \nu_\beta$ oscillations is given by equation (11) of hep-ph/0506165.

If you cannot obtain this result, just take it as a given and proceed.

(b) If the neutrinos are Majorana particles, the neutrino mixing matrix includes two extra Majorana phases, as shown in equation (40) of hep-ph/0506165. Verify that these Majorana phases have no effect on the oscillation probability of equation (11).

Note: Mathematically it is easy to see that this is true. However, since

cancellations do not usually happen by chance, it is useful to try to understand a little better what is going on. Phases can show up in the $\alpha \rightarrow \beta$ probability through the interference of two (or more) paths $\alpha \rightarrow i \rightarrow \beta$ and $\alpha \rightarrow j \rightarrow \beta$. The amplitudes for these two are proportional to $U_{\alpha i}^* U_{\beta i}$ and $U_{\alpha j}^* U_{\beta j}$ respectively, see Figure 1 of hep-ph/0506165. The Majorana phases of equation (40) change $U_{\alpha i} \rightarrow e^{i\rho} U_{\alpha i}$ where the phase ρ depends on i but not on α . Thus in terms like $U_{\alpha i}^* U_{\beta i}$ we pick up a phase factor $e^{-i\rho} e^{i\rho} = 1$, and ρ has no effect on the probability for $\alpha \rightarrow \beta$.

(c) As discussed on page 4 of hep-ph/0506165, there is CP-violation in neutrino oscillations if $\mathcal{I}(U_{\alpha i}^* U_{\beta i} U_{\alpha j} U_{\beta j}^*) \neq 0$ for $\alpha \neq \beta$.

Use unitarity of the neutrino mixing matrix ($\sum U_{\alpha i} U_{\beta i}^* = \delta_{\alpha\beta}$) to show that for a given pair of flavors α and β ($\alpha \neq \beta$), the quantity $\mathcal{I}(U_{\alpha i}^* U_{\beta i} U_{\alpha j} U_{\beta j}^*)$ is the same (up to a sign) for any i and j .

This argument, which is simply based on unitarity, can be expanded by interchanging the roles of the rows and columns of U to conclude that for a given pair of i and j ($i \neq j$) the quantity $\mathcal{I}(U_{\alpha i}^* U_{\beta i} U_{\alpha j} U_{\beta j}^*)$ is independent of α and β (up to a sign).

Thus, $\mathcal{I}(U_{\alpha i}^* U_{\beta i} U_{\alpha j} U_{\beta j}^*)$, on which CP violation depends, is universal (independent of α , β , i , and j).

Use the expression for U given in equation (40) of hep-ph/0506165 (without the Majorana phases) to work out the value of $\mathcal{I}(U_{\alpha i}^* U_{\beta i} U_{\alpha j} U_{\beta j}^*)$ (pick your favorite $\alpha - \beta - i - j$ combination).

Verify that this quantity is proportional to $s_{12}s_{13}s_{23}\sin\delta$. This shows that CP violation needs not only a complex phase δ but also non-trivial mixing between all three neutrino states. In other words, if any of the rotation angles θ_{ij} is equal to 0 or π , the CP violating effects disappear. Note: to save a couple of lines of trivial algebra, you might want to take the mixing matrix from the 2006 PDG, page 138¹, rather than hep-ph/0506165. (The matrix given in the PDG is for quarks rather than neutrinos, but the functional form is identical).

¹Equation 11.2 in <http://pdg.lbl.gov/2006/reviews/kmmixrpp.pdf>